# Finite $XXZ$ critical chain with double boundaries

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**Abstract**

Finite $XXZ$ chain with double boundaries is considered at critical regime $-1 < \Delta < 1$. We construct the eigenvectors of finite Hamiltonian by means of the vertex opertors and the quasi-boundary states. Using the free field realizations of the vertex operators and the quasi-boundary states, integral representations for the correlation functions are derived.

## 1 Introduction

The $XXZ$ chain is a fundamental model in understanding of the integrable systems. Many attentions have been paid to the $XXZ$ integrable systems [1, 2]. The purpose of this paper is to derive correlation functions for finite $XXZ$ chain with double boundaries at critical regime $-1 < \Delta < 1$, by means of the free field approach.

In the earlier works [3, 4] the $XXZ$ chain with a boundary was considered at massive regime $\Delta < -1$, in the framework of the free field approach. The integral representations of the correlation functions were derived. It was shown that boundary quantum

Knizhnik-Zamolodchikov equations with the certain shift, governed the correlation functions. The $U_q(\widehat{sl_n})$-generalization of the papers on the $XXZ$ chain [3, 4] was given in [5, 6]. In the paper [7] results for finite $XXZ$ chain at massive regime $\Delta < -1$ were extended to critical regime $-1 < \Delta < 1$, using bosonizations of vertex operators [8].

Y.Fujii and M.Wadati [9] noticed that solutions of boundary quantm Knizhnik-Zamolodchikov equations without shift became eigenstates of finite $XXZ$ chain with double boundaries. They constructed eigenstates of finite $XXZ$ chain with double boundaries at massive regime $\Delta < -1$, by means of the free field approach. In the present paper we shall study finite $XXZ$ chain with double boundaries at critical regime $-1 < \Delta < 1$. We shall derive the correlation functions as integrals of meromorphic functions involving Multi-Gamma functions.

Now a few words about organiozation of this paper. In section 2 we formulate the problem. In section 3 we construct the realizations of eigenstates. In section 4 we derive integral representations for the correlation functions. In Appendix A we summarized the bosonizations of the vertex operators [8]. In Appendix B we summarized the Multi-Gamma functions.

## 2 Boundary quantum KZ-equation

In 1984 I.V.Cherdnik [10] proposed the following systems of difference equations, now called boundary quantum Knizhnik-Zamolodchikov equations.

$$\begin{aligned} & F(\beta_1, \cdots, \beta_j + i\lambda, \cdots, \beta_N) \\ = \;& T_j(\beta_1, \cdots, \beta_N|\lambda) F(\beta_1, \cdots, \beta_j, \cdots, \beta_N), \; (j = 1, \cdots, N), \end{aligned} \tag{2.1}$$

where the shift operator $T_j(\beta_1 \cdots \beta_N|\lambda)$ is given by

$$\begin{aligned} T_j(\beta_1, \cdots, \beta_N|\lambda) &= R_{j,j-1}(\beta_j - \beta_{j-1} + i\lambda) \cdots R_{j,1}(\beta_j - \beta_1 + i\lambda) \bar{K}_j(\beta_j) \\ &\times R_{1,j}(\beta_1 + \beta_j) \cdots R_{j-1,j}(\beta_{j-1} + \beta_j) R_{j+1,j}(\beta_{j+1} + \beta_j) \cdots R_{N,j}(\beta_N + \beta_j) \\ &\times K_j(\beta_j) R_{j,N}(\beta_j - \beta_N) \cdots R_{j,j+1}(\beta_j - \beta_{j+1}). \end{aligned} \tag{2.2}$$

The $R$-matrix $R(\beta)$ and the boundary $K$-matrix $K(\beta), \bar{K}(\beta)$ are specified later. The solutions of the boundary quantum KZ equations represent various physical quantities. For the case of the shift parameter $\lambda = 2\pi$, the certain solutions of the quantum KZ

equations represents $N$-point correlation functions for the massless $XXZ$ chain with a boundary [7], which is described by the following Hamiltonian :

$$\mathcal{H}=-\frac{1}{2}\sum_{n=1}^{\infty}\left(\sigma_{n+1}^{x}\sigma_{n}^{x}+\sigma_{n+1}^{y}\sigma_{n}^{y}+\Delta\sigma_{n+1}^{z}\sigma_{n}^{z}\right)+h\sigma_{1}^{z}, \tag{2.3}$$

where we set a parameter $-1<\Delta<1$ and a parameter $h$ represents the boundary external field. The $\sigma_n^x,\sigma_n^y$ and $\sigma_n^z$ stand for the Pauli matrices acting on the $n$-th site of the half **Infinite** spin chain : $\cdots\otimes\mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$. The author [7] derived the integrale representations of the correlation functions for the above model.

In the present paper we shall consider the case of the shift parameter $\lambda=0$. In this case the solution of the boundary quantum KZ equation (2.1) represents the eigenvector of finite $XXZ$ chain with double boundaries at critical regime $-1<\Delta<1$. The Hamiltonian $\mathcal{H}_F$ of our considering model is given by

$$\mathcal{H}_F=-\frac{1}{2}\sum_{n=1}^{N-1}\left(\sigma_{n+1}^{x}\sigma_{n}^{x}+\sigma_{n+1}^{y}\sigma_{n}^{y}+\Delta\sigma_{n+1}^{z}\sigma_{n}^{z}\right)+h_1\sigma_{1}^{z}+h_N\sigma_N^z, \tag{2.4}$$

where we set a parameter $-1<\Delta<1$. Parameters $h_1,h_N$ represent the boundary external fields. The $\sigma_n^x,\sigma_n^y$ and $\sigma_n^z$ stand for the Pauli matrices acting on the $n$-th site of the **Finite** spin chain : $(\mathbb{C}^2)^{\otimes N}$.

Let us set the $R$-matrix as

$$R(\beta)=r(\beta)\begin{pmatrix}1&&&\\&b(\beta)&c(\beta)&\\&c(\beta)&b(\beta)&\\&&&1\end{pmatrix}, \tag{2.5}$$

where we set the components as

$$b(\beta)=-\frac{\mathrm{sh}\left(\frac{\beta}{\xi+1}\right)}{\mathrm{sh}\left(\frac{\beta+\pi i}{\xi+1}\right)}\quad,c(\beta)=\frac{\mathrm{sh}\left(\frac{\pi i}{\xi+1}\right)}{\mathrm{sh}\left(\frac{\beta+\pi i}{\xi+1}\right)}. \tag{2.6}$$

Here we set

$$r(\beta)=-\frac{S_2(i\beta|2\pi,\pi(\xi+1))S_2(-i\beta+\pi|2\pi,\pi(\xi+1))}{S_2(-i\beta|2\pi,\pi(\xi+1))S_2(i\beta+\pi|2\pi,\pi(\xi+1))}, \tag{2.7}$$

where $S_2(\beta|\omega_1\omega_2)$ is the double sine function defined in Appendix B.
Let $\{v_+, v_-\}$ denote the natural basis of $V = \mathbb{C}^2$. When viewed as an operator on $V \otimes V$, the matrix elements of $R(\beta)$ are defined by

$$R(\beta) v_{k_1} \otimes v_{k_2} = \sum_{j_1, j_2 = \pm} v_{j_1} \otimes v_{j_2} R(\beta)_{j_1 j_2}^{k_1 k_2}. \tag{2.8}$$

The $R$-matrix satisfies the Yang-Baxter equation :

$$R_{12}(\beta_1 - \beta_2) R_{13}(\beta_1 - \beta_3) R_{23}(\beta_2 - \beta_3) = R_{23}(\beta_2 - \beta_3) R_{13}(\beta_1 - \beta_3) R_{12}(\beta_1 - \beta_2). \tag{2.9}$$

The normalization factor $r_0(\beta)$ is so chosen that the unitarity and crossing relations are

$$\begin{aligned} R_{12}(\beta) R_{21}(-\beta) &= id, &(2.10)\\ R(-\beta)_{j_1 j_2}^{k_1 k_2} &= R(\beta - \pi i)_{-k_2 j_1}^{-j_2 k_1}. &(2.11) \end{aligned}$$

Let us set the boundary $K$-matrix $K(\beta)$ by

$$K(\beta) = k(\beta) \begin{pmatrix} 1 & 0 \\ 0 & \dfrac{\mathrm{sh}\left(\dfrac{\nu + \beta}{\xi + 1}\right)}{\mathrm{sh}\left(\dfrac{\nu - \beta}{\xi + 1}\right)} \end{pmatrix}, \tag{2.12}$$

where the normalization factor is given by

$$k(\beta) = k_0(\beta) k_1(\beta), \tag{2.13}$$

where

$$k_0(\beta) = \frac{S_2(-2i\beta + 4\pi | 4\pi, \pi(\xi+1)) S_2(2i\beta + 3\pi | 4\pi, \pi(\xi+1))}{S_2(2i\beta + 4\pi | 4\pi, \pi(\xi+1)) S_2(-2i\beta + 3\pi | 4\pi, \pi(\xi+1))}, \tag{2.14}$$

$$k_1(\beta) = \frac{S_2(-i\beta + i\nu + \pi | 2\pi, \pi(\xi+1)) S_2(i\beta + i\nu + 2\pi | 2\pi, \pi(\xi+1))}{S_2(i\beta + i\nu + \pi | 2\pi, \pi(\xi+1)) S_2(-i\beta + i\nu + 2\pi | 2\pi, \pi(\xi+1))}. \tag{2.15}$$

The matrix elements $K(\beta)_j^k$ are defined by

$$K(\beta) v_k = \sum_{j = \pm} v_j K(\beta)_j^k. \tag{2.16}$$

The $R$-matrix and the $K$-matrix satisfy the Boundary Yang-Baxter equation.

$$K_2(\beta_2) R_{21}(\beta_1 + \beta_2) K_1(\beta_1) R_{12}(\beta_1 - \beta_2) = R_{21}(\beta_1 - \beta_2) K_1(\beta_1) R_{12}(\beta_1 + \beta_2) K_2(\beta_2) \tag{2.17}$$

The normalization factor $k(\beta)$ is so chosen that the boundary unitarity and the boundary crossing relations are

$$\begin{aligned} K(\beta)K(-\beta) &= id, & (2.18) \\ K\left(\beta+\frac{\pi i}{2}\right)_j^j &= \sum_{k=\pm} R(2\beta)_{k,-k}^{-j,j} K\left(-\beta+\frac{\pi i}{2}\right)_k^k. & (2.19) \end{aligned}$$

Let us set the boundary $K$-matrix $\bar{K}(\beta)$ by

$$\bar{K}(\beta) = K(\beta)|_{\mu\leftrightarrow\nu}. \tag{2.20}$$

*Note. For the another shift parameter* $\lambda = 2\pi$ *case, we take another choice of the* $K$*-matrix* $\bar{K}(\beta)$*. See the reference [7].*

The derivatives of $R$-matrix and $K$-matrix are given by

$$\begin{aligned} & \left.\frac{\partial}{\partial\beta} R_{j,j+1}(\beta) P_{j,j+1}\right|_{\beta=0} \\ = & \frac{-1}{2(\xi+1)} \times \frac{1}{\mathrm{sh}\left(\dfrac{\pi i}{\xi+1}\right)} \left(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y + \Delta \sigma_j^z \sigma_{j+1}^z\right) + const. \end{aligned} \tag{2.21}$$

$$\left.\frac{\partial}{\partial\beta} K_j(\beta)\right|_{\beta=0} = \frac{-1}{\xi+1} \times \frac{\mathrm{ch}\left(\dfrac{\nu}{\xi+1}\right)}{\mathrm{sh}\left(\dfrac{\nu}{\xi+1}\right)} \sigma_j^z + const. \tag{2.22}$$

Here the anisotropic parameter $\Delta = -\cos\left(\frac{\pi}{\xi+1}\right)$.
Therefore the shift operator $T_j(\beta_1 \cdots \beta_N|0)$ is related to the Hamiltonian $\mathcal{H}_F$ (2.4) as following.

$$\mathrm{sh}\left(\frac{\pi i}{\xi+1}\right) \times \frac{\xi+1}{2} \times \left(\frac{\partial}{\partial\beta_j} T_j\right)(0,\cdots,0|0) = \mathcal{H}_F + const. \tag{2.23}$$

Here the boundary magnetic fields $h_1, h_N$ are related with parameters $\mu, \nu$,

$$h_1 = -\frac{1}{2} \times \mathrm{sh}\left(\frac{\pi i}{\xi+1}\right) \frac{\mathrm{ch}\left(\dfrac{\nu}{\xi+1}\right)}{\mathrm{sh}\left(\dfrac{\nu}{\xi+1}\right)}, \quad h_N = -\frac{1}{2} \times \mathrm{sh}\left(\frac{\pi i}{\xi+1}\right) \frac{\mathrm{ch}\left(\dfrac{\mu}{\xi+1}\right)}{\mathrm{sh}\left(\dfrac{\mu}{\xi+1}\right)}. \tag{2.24}$$

We have

$$T_j(0,\cdots,0|0)=id. \tag{2.25}$$

Let us set the eigenvector $|\beta_1\cdots\beta_N\rangle$ by

$$T_j(\beta_1,\cdots,\beta_N|0)|\beta_1,\cdots,\beta_N\rangle=|\beta_1,\cdots,\beta_N\rangle. \tag{2.26}$$

Let us set the dual eigenvector by

$$\langle\beta_1,\cdots,\beta_N|T_j(\beta_1,\cdots,\beta_N|0)=\langle\beta_1,\cdots,\beta_N|. \tag{2.27}$$

The above eigenvectors satisfy the followings.

$$\mathcal{H}_F|0,\cdots,0\rangle=Const.|0,\cdots,0\rangle,\quad \langle 0,\cdots,0|\mathcal{H}_F=Const.\langle 0,\cdots,0|. \tag{2.28}$$

In the next section we shall construct the eigenvector $|\beta_1\cdots\beta_N\rangle$ explicitly.

# 3 Eigenvectors

In this section we solve following eigenvector problem.

$$\mathcal{T}_j(\beta_1,\cdots,\beta_N|0)|\beta_1,\cdots,\beta_N\rangle=|\beta_1,\cdots,\beta_N\rangle. \tag{3.1}$$

The eigenvector is realized by using the vertex operators $\Phi_j(\beta)$.

$$|\beta_1,\cdots,\beta_N\rangle=\frac{1}{\langle G|G\rangle}\sum_{\epsilon_1\cdots\epsilon_N=\pm}\langle G|\Phi_{\epsilon_1}(\beta_1)\cdots\Phi_{\epsilon_N}(\beta_N)|G\rangle(v_{\epsilon_1}\otimes\cdots\otimes v_{\epsilon_N}). \tag{3.2}$$

Here the vector $|G\rangle$ and the dual vector $\langle G|$ are characterized by the following relations.

$$\langle G|\Phi_j(\beta)=\bar{K}(\beta)_j^j\langle G|\Phi_j(-\beta),\ (j=\pm), \tag{3.3}$$

$$\Phi_j(-\beta)|G\rangle=K(\beta)_j^j\Phi_j(\beta)|G\rangle,\ (j=\pm), \tag{3.4}$$

We call the auxiliary states $\langle G|$ and $|G\rangle$ "quasi-boundary state". Using the following commutation relation of the vertex operator and the characterizing relations (3.3, 3.4) of the quasi-boundary state, we have the equation (3.1).

$$\Phi_{j_1}(\beta_1)\Phi_{j_2}(\beta_2)=\sum_{k_1,k_2=\pm}R(\beta_1-\beta_2)_{j_1,j_2}^{k_1,k_2}\Phi_{k_2}(\beta_2)\Phi_{k_1}(\beta_1). \tag{3.5}$$

We give the free field realization of the quasi-boundary state.

Let us introduce the free bosons $b(t), (t \in \mathbb{R})$ by

$$[b(t), b(t')] = \frac{\mathrm{sh}\left(\frac{\pi t}{2}\right)\mathrm{sh}(\pi t)\mathrm{sh}\left(\frac{\pi t\xi}{2}\right)}{t\mathrm{sh}\left(\frac{\pi t(\xi+1)}{2}\right)}\delta(t+t'). \tag{3.6}$$

Let us set the Fock space $\mathcal{H}$ generated by the vacuum vector $\langle vac|$ which satisfies

$$\langle vac|b(-t) = 0, \text{ if } t > 0. \tag{3.7}$$

The quasi-boundary state $\langle G|$ is realized as followings.

$$\langle G| = \langle vac|e^G, \tag{3.8}$$

Here we have set

$$G = \frac{1}{2}\int_0^\infty \frac{G_2(t|\mu)}{[b(t), b(-t)]}b(t)^2 dt + \int_0^\infty \frac{G_1(t|\mu)}{[b(t), b(-t)]}b(t)dt, \tag{3.9}$$

where

$$G_2(t|\mu) = -1, \tag{3.10}$$

$$G_1(t|\mu) = \frac{1}{t}\frac{\mathrm{sh}\left(\frac{\pi}{2}t\right)\mathrm{sh}\left((-i\mu+\frac{\pi}{2}\xi)t\right)}{\mathrm{sh}\left(\frac{\pi}{2}(\xi+1)t\right)} + \frac{1}{t}\frac{\mathrm{sh}\left(\frac{\pi}{4}t\right)\mathrm{sh}\left(\frac{\pi}{2}t\right)\mathrm{ch}\left(\frac{\pi}{4}\xi t\right)}{\mathrm{sh}\left(\frac{\pi}{4}(\xi+1)t\right)}. \tag{3.11}$$

Let us prove the relation (3.3). In what follows we use the abberiviations.

$$U_+(\beta) = \exp\left(-\int_0^\infty \frac{b(t)}{\mathrm{sh}\pi t}e^{i\beta t}dt\right), U_-(\beta) = \exp\left(\int_0^\infty \frac{b(-t)}{\mathrm{sh}\pi t}e^{-i\beta t}dt\right), \tag{3.12}$$

$$\bar{U}_+(\alpha) = \exp\left(\int_0^\infty \frac{b(t)}{\mathrm{sh}\frac{\pi}{2}t}e^{i\alpha t}dt\right), \bar{U}_-(\alpha) = \exp\left(-\int_0^\infty \frac{b(-t)}{\mathrm{sh}\frac{\pi}{2}t}e^{-i\alpha t}dt\right). \tag{3.13}$$

In what follows we omit non-essential constant factors.

At first we explain the formulas of the form

$$X(\beta_1)Y(\beta_2) = C_{XY}(\beta_1-\beta_2) : X(\beta_1)X(\beta_2) :, \tag{3.14}$$

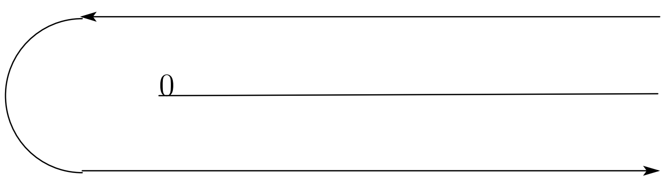
where $X, Y = U_j$, and $C_{XY}(\beta)$ is a meromorphic function on $\mathbb{C}$. These formulae follow from the commutation relation of the free bosons. When we compute the contraction of the basic operators, we often encounter an integral

$$\int_0^\infty F(t)dt, \tag{3.15}$$

which is divergent at $t=0$. Here we adopt the following prescription for regularization : it should be understood as the countour integral,

$$\int_C F(t)\frac{\log(-t)}{2\pi i}dt, \tag{3.16}$$

where the countour $C$ is given by




**Contour** $C$

The actions of the basic oprtaors on quasi-boundary state $\langle G|$ are evaluated as followings.

$$\langle G|U_-(\beta) = Const.m(\beta)\langle G|U_+(-\beta), \tag{3.17}$$

$$\langle G|\bar{U}_-(\alpha) = Const.J(\alpha)\langle G|\bar{U}_+(-\alpha). \tag{3.18}$$

Here we have set

$$\begin{aligned} m(\beta) &= \frac{\Gamma_2(2i\beta+4\pi|2\omega_1\omega_2)\Gamma_2(2i\beta+\pi(\xi+1)|2\omega_1\omega_2)}{\Gamma_2(2i\beta+3\pi|2\omega_1\omega_2)\Gamma_2(2i\beta+\pi(\xi+2)|2\omega_1\omega_2)} \\ &\times \frac{\Gamma_2(i\beta+i\mu+\pi|\omega_1\omega_2)\Gamma_2(i\beta-i\mu+\pi(\xi+2)|\omega_1\omega_2)}{\Gamma_2(i\beta+i\mu+2\pi|\omega_1\omega_2)\Gamma_2(i\beta-i\mu+\pi(\xi+1)|\omega_1\omega_2)}, \end{aligned} \tag{3.19}$$

$$J(\alpha) = \alpha\times\frac{\Gamma\left(\frac{-i\mu+i\alpha}{\pi(\xi+1)}+1-\frac{1}{2(\xi+1)}\right)}{\Gamma\left(\frac{i\mu+i\alpha}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right)}. \tag{3.20}$$

We have

$$\langle G|\Phi_+(\beta) = m(\beta)\langle G|U_+(\beta)U_+(-\beta). \tag{3.21}$$

Because the function $m(\beta)$ satisfies

$$\bar{K}(\beta)_+^+ = \frac{m(\beta)}{m(-\beta)}, \tag{3.22}$$

we have proved the $''+''$-part of the characterizing relation (3.3).

We will prove the $''-''$-part of the equation (3.3). Using the actions formulae of the basic

operators on the quasi-boundary state, we have the following.

$$\begin{aligned}
\langle G|\Phi_-(\beta) &= Const.m(\beta)\int_{-\infty}^{\infty} d\alpha\times\alpha\times\prod_{\epsilon_1,\epsilon_2=\pm}\Gamma\left(\frac{i(\epsilon_1\alpha+\epsilon_2\beta)}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right)\\
&\times \ \mathrm{sh}\left(\frac{\alpha+\beta}{\xi+1}+\frac{\pi i}{2(\xi+1)}\right)\frac{\Gamma\left(\frac{-i\mu+i\alpha}{\pi(\xi+1)}+1-\frac{1}{2(\xi+1)}\right)}{\Gamma\left(\frac{i\mu+i\alpha}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right)}\\
&\times \ \langle G|U_+(\beta)U_+(-\beta)\bar{U}_+(\alpha)\bar{U}_+(-\alpha).
\end{aligned}\tag{3.23}$$

Note that the operator part of the above equation : $\langle G|U_-(\beta)U_-(-\beta)\bar{U}_-(\alpha)\bar{U}_-(-\alpha)$ is invariant under the changes of variables : $\alpha\leftrightarrow-\alpha,\beta\leftrightarrow-\beta$.

We have

$$\begin{aligned}
& m(\beta)^{-1}\mathrm{sh}\left(\frac{\mu-\beta}{\xi+1}\right)\langle G|\Phi_-(\beta)-m(-\beta)^{-1}\mathrm{sh}\left(\frac{\mu+\beta}{\xi+1}\right)\langle G|\Phi_-(-\beta)\\
= \ & Const.\times\mathrm{sh}\left(\frac{2\beta}{\xi+1}\right)\int_{-\infty}^{\infty}d\alpha\prod_{\epsilon_1,\epsilon_2=\pm}\Gamma\left(\frac{i(\epsilon_1\alpha+\epsilon_2\beta)}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right)\\
\times \ & \prod_{\epsilon=\pm}\Gamma\left(\frac{i(-\mu+\epsilon\alpha)}{\pi(\xi+1)}+1-\frac{1}{2(\xi+1)}\right)\times\alpha\prod_{\epsilon=\pm}\mathrm{sh}\left(\frac{\mu+\epsilon\alpha}{\xi+1}-\frac{\pi i}{2(\xi+1)}\right)\\
\times \ & \langle G|U_-(\beta)U_-(-\beta)\bar{U}_-(\alpha)\bar{U}_-(-\alpha).
\end{aligned}\tag{3.24}$$

The integrand of (RHS) is anti-symmetric to a change of integral variable $\alpha\leftrightarrow-\alpha$. It means the left-hand side becomes zero after taking integral. Therefore we get

$$m(-\beta)\mathrm{sh}\left(\frac{\mu-\beta}{\xi+1}\right)\langle G|\Phi_-(\beta)=m(\beta)\mathrm{sh}\left(\frac{\mu+\beta}{\xi+1}\right)\langle G|\Phi_-(-\beta).\tag{3.25}$$

We have proved the $''-''$-part of the characterizing relation (3.3).

The quasi-boundary state $|G\rangle$ is given by the following.

$$|G\rangle=e^{G*}|vac\rangle,\tag{3.26}$$

where

$$G^*=\frac{1}{2}\int_0^\infty\frac{G_2^*(t|\mu)}{[b(t),b(-t)]}b(-t)^2dt+\int_0^\infty\frac{G_1^*(t|\mu)}{[b(t),b(-t)]}b(-t)dt,\tag{3.27}$$

where

$$G_2^*(t|\mu)=G_2(t|\mu),\ G_1^*(t|\mu)=-G_1(t|\mu).\tag{3.28}$$

As the same manner as the above we can prove the characterizing equations (3.4).

*Note.* *When we consider the massless XXZ chain with a boundary [7]. We introduce the boundary state $|B\rangle$ and the dual boundary state $\langle B|$. Left quasi-boundary state $\langle G|$ differes from the dual boundary state $\langle B|$. Right quasi-boundary state $|G\rangle$ coincides with the boundary state $|B\rangle$. Physically the boundary state $|B\rangle$ of the paper [7] coresponds to the vacuum expectation value $\langle G|\Phi(\beta_1)\cdots\Phi(\beta_N)|G\rangle$ of the present paper. Both quantities $|B\rangle$ and $\langle G|\Phi(\beta_1)\cdots\Phi(\beta_N)|G\rangle$ represent an eigenvector of the Hamiltonian for each model.*

Let us construct the dual stationary state,

$$\langle\beta_1,\cdots\beta_N|T_j(\beta_1,\cdots,\beta_N|0)=\langle\beta_1,\cdots\beta_N|. \tag{3.29}$$

The dual stationary state is realized by using the dual vertex operators $\Phi_j^*(\beta)$.

$$\langle\beta_1,\cdots\beta_N|=\frac{1}{\langle F|F\rangle}\sum_{\epsilon_1,\cdots,\epsilon_N=\pm}\langle F|\Phi_{\epsilon_1}^*(\beta_1)\cdots\Phi_{\epsilon_N}^*(\beta_N)|F\rangle(v_{\epsilon_1}^*\otimes\cdots\otimes v_{\epsilon_N}^*). \tag{3.30}$$

The dual vertex operators are related to the vertex operators,

$$\Phi_j^*(\beta)=\Phi_{-j}(\beta+\pi i),\ (j=\pm). \tag{3.31}$$

The quasi-boundary state $\langle F|$ and $|F\rangle$ are characterized by

$$\langle F|\Phi_j^*(\beta)=K(\beta)_j^j\langle F|\Phi_j^*(-\beta),\ (j=\pm), \tag{3.32}$$

$$\Phi_j^*(-\beta)|F\rangle=\bar{K}(\beta)_j^j\Phi_j^*(\beta)|F\rangle,\ (j=\pm). \tag{3.33}$$

The quasi-boundary states $\langle F|$ and $|F\rangle$ are realized as followings.

$$\langle F|=\langle vac|e^F,\quad |F\rangle=e^{F*}|vac\rangle. \tag{3.34}$$

Here we have set

$$F = \frac{1}{2}\int_0^\infty\frac{F_2(t|\mu)}{[b(t),b(-t)]}b(t)^2dt+\int_0^\infty\frac{F_1(t|\mu)}{[b(t),b(-t)]}b(t)dt, \tag{3.35}$$

$$F^* = \frac{1}{2}\int_0^\infty\frac{F_2^*(t|\nu)}{[b(t),b(-t)]}b(-t)^2dt+\int_0^\infty\frac{F_1^*(t|\nu)}{[b(t),b(-t)]}b(-t)dt, \tag{3.36}$$

where

$$F_2(t|\mu) = -e^{-2\pi t}, \tag{3.37}$$

$$F_1(t|\mu) = \frac{e^{-\pi t}}{t}\frac{\mathrm{sh}\left(\frac{\pi t}{2}\right)\mathrm{sh}\left((i\mu-\frac{\pi\xi}{2}-\pi)t\right)}{\mathrm{sh}\left(\frac{\pi}{2}(\xi+1)t\right)}+\frac{e^{-\pi t}}{t}\frac{\mathrm{sh}\left(\frac{\pi t}{2}\right)\mathrm{sh}\left(\frac{\pi t}{4}\right)\mathrm{ch}\left(\frac{\pi}{4}\xi t\right)}{\mathrm{sh}\left(\frac{\pi}{4}(\xi+1)t\right)}. \tag{3.38}$$

$$F_2^*(t|\nu) = -e^{2\pi t}, \tag{3.39}$$

$$F_1^*(t|\nu) = -e^{2\pi t} \times F_1(t|\nu). \tag{3.40}$$

The characterizing relations are proved as the same manner. Here we omit details. Formally other eigenvectors are constructed by inserting the type-II vertex operators.

$$\sum_{\epsilon_1 \cdots \epsilon_N = \pm} \langle G | \Phi_{\epsilon_1}(\beta_1) \cdots \Phi_{\epsilon_N}(\beta_N) \Psi^*_{j_1}(\xi_1) \cdots \Psi^*_{j_M}(\xi_M) | G \rangle (v_{\epsilon_1} \otimes \cdots \otimes v_{\epsilon_N}). \tag{3.41}$$

# 4 Correlation Functions

In this section we calculate the vacuum expectation values of the type-I vertex operators, and obtain them as integrals of meromorphic functions involving Multi-Gamma functions. We compute the following $2N$-point function,

$$\begin{aligned} & P_{\epsilon_1^*, \cdots, \epsilon_N^*; \epsilon_N, \cdots, \epsilon_1; \eta}(\{\beta_j^*\}; \{\beta_j\}) \\ = & \frac{\langle F_\eta | \Phi_{\epsilon_1^*}(\beta_1^*) \cdots \Phi_{\epsilon_N^*}(\beta_N^*) | F_\eta \rangle}{\langle F_\eta | F_\eta \rangle} \times \frac{\langle G_\eta | \Phi_{\epsilon_N}(\beta_N) \cdots \Phi_{\epsilon_1}(\beta_1) | G_\eta \rangle}{\langle G_\eta | G_\eta \rangle}. \end{aligned} \tag{4.1}$$

Here we set the state $\langle G_\eta |, | G_\eta \rangle$ and $\langle F_\eta |, | F_\eta \rangle$ by

$$\langle G_\eta | = \langle vac | e^{G_\eta}, \quad | G_\eta \rangle = e^{G_\eta^*} | vac \rangle, \tag{4.2}$$

$$\langle F_\eta | = \langle vac | e^{F_\eta}, \quad | F_\eta \rangle = e^{F_\eta^*} | vac \rangle. \tag{4.3}$$

where

$$G_\eta = \frac{1}{2} \int_0^\infty \frac{e^{-\eta t} G_2(t|\mu))}{[b(t), b(-t)]} b(t)^2 dt + \int_0^\infty \frac{G_1(t|\mu)}{[b(t), b(-t)]} b(t) dt, \tag{4.4}$$

$$G_\eta^* = \frac{1}{2} \int_0^\infty \frac{e^{-\eta t} G_2^*(t|\mu))}{[b(t), b(-t)]} b(-t)^2 dt + \int_0^\infty \frac{G_1^*(t|\mu)}{[b(t), b(-t)]} b(-t) dt \tag{4.5}$$

$$F_\eta = \frac{1}{2} \int_0^\infty \frac{e^{-\eta t} F_2(t|\nu)}{[b(t), b(-t)]} b(t)^2 dt + \int_0^\infty \frac{F_1(t|\nu)}{[b(t), b(-t)]} b(t) dt, \tag{4.6}$$

$$F_\eta^* = \frac{1}{2} \int_0^\infty \frac{e^{-\eta t} F_2^*(t|\nu)}{[b(t), b(-t)]} b(-t)^2 dt + \int_0^\infty \frac{F_1^*(t|\nu)}{[b(t), b(-t)]} b(-t) dt. \tag{4.7}$$

We have

$$\lim_{\eta \to 0} \langle G_\eta | = \langle G |, \quad \lim_{\eta \to 0} | G_\eta \rangle = | G \rangle, \quad \lim_{\eta \to 0} \langle F_\eta | = \langle F |, \quad \lim_{\eta \to 0} | F_\eta \rangle = | F \rangle. \tag{4.8}$$

First we compute the normal-ordering of the vertex operators.

Let us denote by $A$ the index set

$$A = \{ a | \epsilon_a = -, 1 \le a \le N \} \tag{4.9}$$

We have the following expressions.

$$
\begin{aligned}
&\frac{\langle G_\eta|\Phi_{\epsilon_N}(\beta_N)\cdots\Phi_{\epsilon_1}(\beta_1)|G_\eta\rangle}{\langle G_\eta|G_\eta\rangle}\\
=&\prod_{1\le b_2<b_1\le N}\frac{\Gamma_2\left(i(\beta_{b_2}-\beta_{b_1})+2\pi|\omega_1\omega_2\right)\Gamma_2\left(i(\beta_{b_2}-\beta_{b_1})+\pi(\xi+1)|\omega_1\omega_2\right)}{\Gamma_2\left(i(\beta_{b_2}-\beta_{b_1})+\pi|\omega_1\omega_2\right)\Gamma_2\left(i(\beta_{b_2}-\beta_{b_1})+\pi(\xi+2)|\omega_1\omega_2\right)}\\
\times&\prod_{a\in A}\int_{-\infty}^{\infty}d\alpha_a\prod_{a\in A}\Gamma\left(\frac{i(\alpha_a-\beta_a)}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right)\Gamma\left(\frac{i(\beta_a-\alpha_a)}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right)\\
\times&\prod_{\substack{a_2<a_1\\ a_1,a_2\in A}}\left\{(\alpha_{a_2}-\alpha_{a_1})\frac{\Gamma\left(\frac{i(\alpha_{a_2}-\alpha_{a_1})}{\pi(\xi+1)}+1-\frac{1}{\xi+1}\right)}{\Gamma\left(\frac{i(\alpha_{a_2}-\alpha_{a_1})}{\pi(\xi+1)}+\frac{1}{\xi+1}\right)}\right\}\\
\times&\prod_{\substack{a>b\\ a\in A,1\le b\le N}}\frac{\Gamma\left(\frac{i(\beta_b-\alpha_a)}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right)}{\Gamma\left(\frac{i(\beta_b-\alpha_a)}{\pi(\xi+1)}+1-\frac{1}{2(\xi+1)}\right)}\prod_{\substack{b>a\\ a\in A,1\le b\le N}}\frac{\Gamma\left(\frac{i(\alpha_a-\beta_b)}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right)}{\Gamma\left(\frac{i(\alpha_a-\beta_b)}{\pi(\xi+1)}+1-\frac{1}{2(\xi+1)}\right)}\\
\times&\ I_\eta(\{\beta_b\}|\{\alpha_a\}).
\end{aligned}
\tag{4.10}
$$

Let us denote by $A^*$ the undex set

$$
A^*=\{a|\epsilon_a^*=+,1\le a\le N\}
\tag{4.11}
$$

We have the following expressions.

$$
\begin{aligned}
&\frac{\langle F_\eta|\Phi^*_{\epsilon_1*}(\beta_1^*-\pi i)\cdots\Phi_{\epsilon_N*}(\beta_N^*-\pi i)|F_\eta\rangle}{\langle F_\eta|F_\eta\rangle}\\
=&\prod_{1\le b_1<b_2\le N}\frac{\Gamma_2\left(i(\beta_{b_2}-\beta_{b_1})+2\pi|\omega_1\omega_2\right)\Gamma_2\left(i(\beta_{b_2}-\beta_{b_1})+\pi(\xi+1)|\omega_1\omega_2\right)}{\Gamma_2\left(i(\beta_{b_2}-\beta_{b_1})+\pi|\omega_1\omega_2\right)\Gamma_2\left(i(\beta_{b_2}-\beta_{b_1})+\pi(\xi+2)|\omega_1\omega_2\right)}\\
\times&\prod_{a\in A^*}\int_{-\infty}^{\infty}d\alpha_a\prod_{a\in A^*}\Gamma\left(\frac{i(\alpha_a-\beta_a)}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right)\Gamma\left(\frac{i(\beta_a-\alpha_a)}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right)\\
\times&\prod_{\substack{a_1<a_2\\ a_1,a_2\in A^*}}\left\{(\alpha_{a_2}-\alpha_{a_1})\frac{\Gamma\left(\frac{i(\alpha_{a_2}-\alpha_{a_1})}{\pi(\xi+1)}+1-\frac{1}{\xi+1}\right)}{\Gamma\left(\frac{i(\alpha_{a_2}-\alpha_{a_1})}{\pi(\xi+1)}+\frac{1}{\xi+1}\right)}\right\}\\
\times&\prod_{\substack{a<b\\ a\in A^*,1\le b\le N}}\frac{\Gamma\left(\frac{i(\beta_b-\alpha_a)}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right)}{\Gamma\left(\frac{i(\beta_b-\alpha_a)}{\pi(\xi+1)}+1-\frac{1}{2(\xi+1)}\right)}\prod_{\substack{b<a\\ a\in A^*,1\le b\le N}}\frac{\Gamma\left(\frac{i(\alpha_a-\beta_b)}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right)}{\Gamma\left(\frac{i(\alpha_a-\beta_b)}{\pi(\xi+1)}+1-\frac{1}{2(\xi+1)}\right)}\\
\times&\ I^*_\eta(\{\beta_b^*-\pi i\}|\{\alpha_a\}).
\end{aligned}
\tag{4.12}
$$

Here we have set

$$I_\eta(\{\beta_b\}|\{\alpha_a\}) = \frac{\langle G_\eta| \exp\left(\int_0^\infty X_A(t)b(-t)dt\right) \exp\left(\int_0^\infty Y_A(t)b(t)dt\right)|G_\eta\rangle}{\langle G_\eta|G_\eta\rangle}, \quad (4.13)$$

$$I_\eta^*(\{\beta_b^* - \pi i\}|\{\alpha_a^\}) = \frac{\langle F_\eta| \exp\left(\int_0^\infty X_{A^*}(t)b(-t)dt\right) \exp\left(\int_0^\infty Y_{A^*}(t)b(t)dt\right)|F_\eta\rangle}{\langle F_\eta|F_\eta\rangle}. \quad (4.14)$$

with

$$X_A(t) = \sum_{b=1}^N \frac{e^{-i\beta_b t}}{\mathrm{sh}(\pi t)} - \sum_{a\in A} \frac{e^{-i\alpha_a t}}{\mathrm{sh}\left(\frac{\pi t}{2}\right)}, \quad (4.15)$$

$$Y_A(t) = -\sum_{b=1}^N \frac{e^{i\beta_b t}}{\mathrm{sh}(\pi t)} + \sum_{a\in A} \frac{e^{i\alpha_a t}}{\mathrm{sh}\left(\frac{\pi t}{2}\right)}. \quad (4.16)$$

We evaluate the quantities $I_\eta(\{\beta_b\}|\{\alpha_a\}), I_\eta^*(\{\beta_b^* - \pi i\}|\{\alpha_a\})$. Using the completeness relation of the coherent state [3, 7], and performing the integral calculations, we have

$$\begin{aligned} & I_\eta(\{\beta_b\}|\{\alpha_a\}) \\ = \; & \exp\left(\int_0^\infty \frac{1}{1-e^{-2\eta t}} \frac{\mathrm{sh}\left(\frac{\pi t}{2}\right)\mathrm{sh}\left(\pi t\right)\mathrm{sh}\left(\frac{\pi \xi t}{2}\right)}{t\mathrm{sh}\left(\frac{\pi}{2}(\xi+1)t\right)} \right. \\ \times \; & \left(-\frac{1}{2}e^{-\eta t}X_A(t)^2 + e^{-2\eta t}X_A(t)Y_A(t) - \frac{1}{2}e^{-\eta t}Y_A(t)^2\right) \\ + \; & \left. \int_0^\infty \frac{1}{1-e^{-2\eta t}}\{(G_1(t|\mu) - e^{-\eta t}G_1^*(t|\nu))X_A(t) + (G_1^*(t|\nu) - e^{-\eta t}G_1(t|\mu))Y_A(t)\}dt\right). \end{aligned} \quad (4.17)$$

and

$$\begin{aligned} & I_\eta^*(\{\beta_b^* - \pi i\}|\{\alpha_a\}) \\ = \; & \exp\left(\int_0^\infty \frac{1}{1-e^{-2\eta t}} \frac{\mathrm{sh}\left(\frac{\pi t}{2}\right)\mathrm{sh}\left(\pi t\right)\mathrm{sh}\left(\frac{\pi \xi t}{2}\right)}{t\mathrm{sh}\left(\frac{\pi}{2}(\xi+1)t\right)} \right. \\ \times \; & \left(-\frac{1}{2}e^{-\eta t-2\pi t}X_{A^*}(t)^2 + e^{-2\eta t}X_{A^*}(t)Y_{A^*}(t) - \frac{1}{2}e^{-\eta t+2\pi t}Y_{A^*}(t)^2\right) \\ + \; & \left. \int_0^\infty \frac{1}{1-e^{-2\eta t}}\{(F_1(t|\mu) - e^{-\eta t-2\pi t}F_1^*(t|\nu))X_{A^*}(t) + (F_1^*(t|\nu) - e^{-\eta t+2\pi t}F_1(t|\mu))Y_{A^*}(t)\}dt\right). \end{aligned} \quad (4.18)$$

In what follows we use the abberiviations :

$$\omega_1 = 2\pi, \omega_2 = \pi(\xi+1), \omega_3 = 2\eta, \quad \mu_+ = \mu, \mu_- = \nu, \quad (4.19)$$

The vacuum expectation value is evaluated as following.

$$I_\eta(\{\beta_b\}|\{\alpha_a\}) = I_\eta^\beta(\{\beta_b\})I_\eta^{\beta\alpha}(\{\beta_b\}|\{\alpha_a\})I_\eta^\alpha(\{\alpha_a\}). \quad (4.20)$$

Here we set

$$
\begin{aligned}
& I_\eta^\beta(\{\beta_b\}) \\
= & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\sqrt{\frac{S_3(2i\epsilon\beta_b+\pi+\eta|\omega_1\omega_2\omega_3)}{S_3(2i\epsilon\beta_b+2\pi+\eta|\omega_1\omega_2\omega_3)}}\prod_{b_1<b_2}\prod_{\epsilon=\pm}\frac{\Gamma_2(i\epsilon(\beta_{b_1}-\beta_{b_2})+\pi|\omega_1\omega_2)}{\Gamma_2(i\epsilon(\beta_{b_1}-\beta_{b_2})+2\pi|\omega_1\omega_2)} \\
\times & \prod_{b_1<b_2}\prod_{\epsilon=\pm}\frac{S_3(i\epsilon(\beta_{b_1}+\beta_{b_2})+\pi+\eta|\omega_1\omega_2\omega_3)S_3(i\epsilon(\beta_{b_1}-\beta_{b_2})+\pi|\omega_1\omega_2\omega_3)}{S_3(i\epsilon(\beta_{b_1}+\beta_{b_2})+2\pi+\eta|\omega_1\omega_2\omega_3)S_3(i\epsilon(\beta_{b_1}-\beta_{b_2})+2\pi|\omega_1\omega_2\omega_3)} \\
\times & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\frac{\Gamma_3(i\epsilon\beta_b+i\mu_\epsilon+\pi|\omega_1\omega_2\omega_3)\Gamma_3(i\epsilon\beta_b-i\mu_\epsilon+\pi\xi+2\pi|\omega_1\omega_2\omega_3)}{\Gamma_3(i\epsilon\beta_b-i\mu_\epsilon+\pi\xi+\pi|\omega_1\omega_2\omega_3)\Gamma_3(i\epsilon\beta_b+i\mu_\epsilon+2\pi|\omega_1\omega_2\omega_3)} \\
\times & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\frac{\Gamma_3(-i\epsilon\beta_b+\eta+i\mu_\epsilon+\pi|\omega_1\omega_2\omega_3)\Gamma_3(-i\epsilon\beta_b+\eta-i\mu_\epsilon+\pi\xi+2\pi|\omega_1\omega_2\omega_3)}{\Gamma_3(-i\epsilon\beta_b+\eta-i\mu_\epsilon+\pi\xi+\pi|\omega_1\omega_2\omega_3)\Gamma_3(-i\epsilon\beta_b+\eta+i\mu_\epsilon+2\pi|\omega_1\omega_2\omega_3)} \\
\times & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\sqrt{\frac{\Gamma_3(2i\epsilon\beta_b+\pi|2\omega_1,\omega_2,2\omega_3)\Gamma_3(2i\epsilon\beta_b+4\pi|2\omega_1,\omega_2,2\omega_3)}{\Gamma_3(2i\epsilon\beta_b+2\pi|2\omega_1,\omega_2,2\omega_3)\Gamma_3(2i\epsilon\beta_b+3\pi|2\omega_1,\omega_2,2\omega_3)}} \\
\times & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\sqrt{\frac{\Gamma_3(2i\epsilon\beta_b+\pi\xi+\pi|2\omega_1,\omega_2,2\omega_3)\Gamma_3(2i\epsilon\beta_b+\pi\xi+4\pi|2\omega_1,\omega_2,2\omega_3)}{\Gamma_3(2i\epsilon\beta_b+\pi\xi+2\pi|2\omega_1,\omega_2,2\omega_3)\Gamma_3(2i\epsilon\beta_b+\pi\xi+3\pi|2\omega_1,\omega_2,2\omega_3)}} \\
\times & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\sqrt{\frac{\Gamma_3(2i\epsilon\beta_b+2\eta+\pi|2\omega_1,\omega_2,2\omega_3)\Gamma_3(2i\epsilon\beta_b+2\eta+4\pi|2\omega_1,\omega_2,2\omega_3)}{\Gamma_3(2i\epsilon\beta_b+2\eta+2\pi|2\omega_1,\omega_2,2\omega_3)\Gamma_3(2i\epsilon\beta_b+2\eta+3\pi|2\omega_1,\omega_2,2\omega_3)}} \\
\times & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\sqrt{\frac{\Gamma_3(2i\epsilon\beta_b+2\eta+\pi\xi+\pi|2\omega_1,\omega_2,2\omega_3)\Gamma_3(2i\epsilon\beta_b+2\eta+\pi\xi+4\pi|2\omega_1,\omega_2,2\omega_3)}{\Gamma_3(2i\epsilon\beta_b+2\eta+\pi\xi+2\pi|2\omega_1,\omega_2,2\omega_3)\Gamma_3(2i\epsilon\beta_b+2\eta+\pi\xi+3\pi|2\omega_1,\omega_2,2\omega_3)}}
\end{aligned}
\tag{4.21}
$$

$$
\begin{aligned}
& I_\eta^\alpha(\{\alpha_a\}) \\
= & \prod_{a\in A}\prod_{\epsilon=\pm}\sqrt{\frac{S_2(2i\epsilon\alpha_a+\eta|\omega_2\omega_3)}{S_2(2i\epsilon\alpha_a+\pi\xi+\eta|\omega_2\omega_3)}}\prod_{a_1<a_2}\prod_{\epsilon=\pm}\frac{\Gamma\left(\frac{i(\alpha_{a_1}-\alpha_{a_2})}{\pi(\xi+1)}+\frac{1}{\xi+1}\right)}{\Gamma\left(\frac{i(\alpha_{a_1}-\alpha_{a_2})}{\pi(\xi+1)}+1\right)} \\
\times & \prod_{a_1<a_2}\prod_{\epsilon=\pm}\frac{S_2(i\epsilon(\alpha_{a_1}+\alpha_{a_2})+\eta|\omega_2\omega_3)S_2(i\epsilon(\alpha_{a_1}-\alpha_{a_2})+\pi|\omega_2\omega_3)}{S_2(i\epsilon(\alpha_{a_1}+\alpha_{a_2})+\pi\xi+\eta|\omega_2\omega_3)S_2(i\epsilon(\alpha_{a_1}-\alpha_{a_2})+\pi(\xi+1)|\omega_2\omega_3)} \\
\times & \prod_{a\in A}\prod_{\epsilon=\pm}\frac{\Gamma_2(i\epsilon\alpha_a+\pi\xi+\frac{\pi}{2}-i\mu_\epsilon|\omega_2\omega_3)\Gamma_2(-i\epsilon\alpha_a+\eta+\pi\xi+\frac{\pi}{2}-i\mu_\epsilon|\omega_2\omega_3)}{\Gamma_2(i\epsilon\alpha_a+\frac{\pi}{2}+i\mu_\epsilon|\omega_2\omega_3)\Gamma_2(-i\epsilon\alpha_a+\eta+\frac{\pi}{2}+i\mu_\epsilon|\omega_2\omega_3)} \\
\times & \prod_{a\in A}\prod_{\epsilon=\pm}\sqrt{\frac{\Gamma_2(2i\epsilon\alpha_a+\pi|\omega_2,2\omega_3)\Gamma_2(2i\epsilon\alpha_a+\pi(\xi+1)|\omega_2,2\omega_3)}{\Gamma_2(2i\epsilon\alpha_a|\omega_2,2\omega_3)\Gamma_2(2i\epsilon\alpha_a+\pi\xi|\omega_2,2\omega_3)}} \\
\times & \prod_{a\in A}\prod_{\epsilon=\pm}\sqrt{\frac{\Gamma_2(2i\epsilon\alpha_a+2\eta+\pi|\omega_2,2\omega_3)\Gamma_2(2i\epsilon\alpha_a+2\eta+\pi(\xi+1)|\omega_2,2\omega_3)}{\Gamma_2(2i\epsilon\alpha_a+2\eta|\omega_2,2\omega_3)\Gamma_2(2i\epsilon\alpha_a+2\eta+\pi\xi|\omega_2,2\omega_3)}}.
\end{aligned}
\tag{4.22}
$$

$$
\begin{aligned}
& I_\eta^{\beta\alpha}(\{\beta_b\}|\{\alpha_a\}) \\
= & \prod_{a\in A}\prod_{b=1}^{N}\frac{S_2(i(\alpha_a+\beta_b)+\eta+\pi\xi+\frac{\pi}{2}|\omega_2\omega_3)}{S_2(i(\alpha_a+\beta_b)+\eta+\frac{\pi}{2}|\omega_2\omega_3)} \\
\times & \prod_{a\in A}\prod_{b=1}^{N}\prod_{\epsilon=\pm}\left\{\Gamma\left(\frac{i\epsilon(\alpha_a-\beta_b)}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right)S_2\left(i\epsilon(\alpha_a-\beta_b)+\frac{\pi}{2}\middle|\,\omega_2\omega_3\right)\right\}^{-1}.
\end{aligned}
\tag{4.23}
$$

The vacuum expectation value is evaluated as following.

$$
I_\eta^*(\{\beta_b^*-\pi i\}|\{\alpha_a\}) = I_\eta^{*\beta}(\{\beta_b^*\})I_\eta^{*\beta\alpha}(\{\beta_b^*\}|\{\alpha_a\})I_\eta^{*\alpha}(\{\alpha_a\}). \tag{4.24}
$$

Here we set

$$
\begin{aligned}
& I_\eta^{*\beta}(\{\beta_b^*\}) \\
= & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\sqrt{\frac{S_3(2i\epsilon\beta_b^*+\pi+2\pi\epsilon+\eta|\omega_1\omega_2\omega_3)}{S_3(2i\epsilon\beta_b^*+2\pi+2\pi\epsilon+\eta|\omega_1\omega_2\omega_3)}}\prod_{b_1<b_2}\prod_{\epsilon=\pm}\frac{\Gamma_2(i\epsilon(\beta_{b_1}^*-\beta_{b_2}^*)+\pi|\omega_1\omega_2)}{\Gamma_2(i\epsilon(\beta_{b_1}^*-\beta_{b_2}^*)+2\pi|\omega_1\omega_2)} \\
\times & \prod_{b_1<b_2}\prod_{\epsilon=\pm}\frac{S_3(i\epsilon(\beta_{b_1}^*+\beta_{b_2}^*)+\pi+2\pi\epsilon+\eta|\omega_1\omega_2\omega_3)S_3(i\epsilon(\beta_{b_1}^*-\beta_{b_2}^*)+\pi|\omega_1\omega_2\omega_3)}{S_3(i\epsilon(\beta_{b_1}^*+\beta_{b_2}^*)+2\pi+2\pi\epsilon+\eta|\omega_1\omega_2\omega_3)S_3(i\epsilon(\beta_{b_1}^*-\beta_{b_2}^*)+2\pi|\omega_1\omega_2\omega_3)} \\
\times & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\frac{\Gamma_3(i\epsilon\beta_b^*-i\mu_\epsilon+\pi\xi+2\pi+\pi\epsilon|\omega_1\omega_2\omega_3)\Gamma_3(i\epsilon\beta_b^*+i\mu_\epsilon+\pi\epsilon+\pi|\omega_1\omega_2\omega_3)}{\Gamma_3(i\epsilon\beta_b^*+i\mu_\epsilon+\pi\epsilon|\omega_1\omega_2\omega_3)\Gamma_3(i\epsilon\beta_b^*-i\mu_\epsilon+\pi\xi+3\pi+\pi\epsilon|\omega_1\omega_2\omega_3)} \\
\times & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\frac{\Gamma_3(-i\epsilon\beta_b^*-i\mu_\epsilon+\eta+\pi\xi+2\pi-\pi\epsilon|\omega_1\omega_2\omega_3)\Gamma_3(-i\epsilon\beta_b^*+\eta+i\mu_\epsilon+\pi-\pi\epsilon|\omega_1\omega_2\omega_3)}{\Gamma_3(-i\epsilon\beta_b^*+i\mu_\epsilon+\eta-\pi\epsilon|\omega_1\omega_2\omega_3)\Gamma_3(-i\epsilon\beta_b^*-i\mu_\epsilon+\eta+\pi\xi+\pi-\pi\epsilon|\omega_1\omega_2\omega_3)} \\
\times & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\sqrt{\frac{\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+\pi|2\omega_1,\omega_2,2\omega_3)\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+4\pi|2\omega_1,\omega_2,2\omega_3)}{\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+2\pi|2\omega_1,\omega_2,2\omega_3)\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+3\pi|2\omega_1,\omega_2,2\omega_3)}} \\
\times & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\sqrt{\frac{\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+\pi\xi+\pi|2\omega_1,\omega_2,2\omega_3)\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+\pi\xi+4\pi|2\omega_1,\omega_2,2\omega_3)}{\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+\pi\xi+2\pi|2\omega_1,\omega_2,2\omega_3)\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+\pi\xi+3\pi|2\omega_1,\omega_2,2\omega_3)}} \\
\times & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\sqrt{\frac{\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+2\eta+\pi|2\omega_1,\omega_2,2\omega_3)\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+2\eta+4\pi|2\omega_1,\omega_2,2\omega_3)}{\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+2\eta+2\pi|2\omega_1,\omega_2,2\omega_3)\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+2\eta+3\pi|2\omega_1,\omega_2,2\omega_3)}} \\
\times & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\sqrt{\frac{\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+2\eta+\pi\xi+\pi|2\omega_1,\omega_2,2\omega_3)}{\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+2\eta+\pi\xi+2\pi|2\omega_1,\omega_2,2\omega_3)}} \\
\times & \prod_{b=1}^{N}\prod_{\epsilon=\pm}\sqrt{\frac{\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+2\eta+\pi\xi+4\pi|2\omega_1,\omega_2,2\omega_3)}{\Gamma_3(2i\epsilon\beta_b^*+2\pi\epsilon+2\eta+\pi\xi+3\pi|2\omega_1,\omega_2,2\omega_3)}}.
\end{aligned}
\tag{4.25}
$$

$$
\begin{aligned}
&I_\eta^{*\beta\alpha}(\{\beta_b^*\}|\{\alpha_a\}) \\
= \ & \prod_{a\in A^*}\prod_{b=1}^{N} \frac{S_2(i(\alpha_a+\beta_b^*)+\eta+\pi\xi-\frac{3\pi}{2}|\omega_2\omega_3)}{S_2(i(\alpha_a+\beta_b^*)+\eta-\frac{3\pi}{2}|\omega_2\omega_3)} \\
\times \ & \prod_{a\in A^*}\prod_{b=1}^{N}\prod_{\epsilon=\pm}\left\{\Gamma\left(\frac{i\epsilon(\alpha_a-\beta_b^*)}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right) S_2\left(i\epsilon(\alpha_a-\beta_b^*)+\frac{\pi}{2}\middle|\,\omega_2\omega_3\right)\right\}^{-1}.
\end{aligned}
\tag{4.26}
$$

$$
\begin{aligned}
&I_\eta^{*\alpha}(\{\alpha_a\}) \\
= \ & \prod_{a\in A^*}\prod_{\epsilon=\pm}\sqrt{\frac{S_2(2i\epsilon\alpha_a+2\pi\epsilon+\eta|\omega_2\omega_3)}{S_2(2i\epsilon\alpha_a+\pi\xi+2\pi\epsilon+\eta|\omega_2\omega_3)}}\prod_{a_1<a_2}\prod_{\epsilon=\pm}\frac{\Gamma\left(\frac{i(\alpha_{a_1}-\alpha_{a_2})}{\pi(\xi+1)}+\frac{1}{\xi+1}\right)}{\Gamma\left(\frac{i(\alpha_{a_1}-\alpha_{a_2})}{\pi(\xi+1)}+1\right)} \\
\times \ & \prod_{a_1<a_2}\prod_{\epsilon=\pm}\frac{S_2(i\epsilon(\alpha_{a_1}+\alpha_{a_2})+2\pi\epsilon+\eta|\omega_2\omega_3)S_2(i\epsilon(\alpha_{a_1}-\alpha_{a_2})+\pi|\omega_2\omega_3)}{S_2(i\epsilon(\alpha_{a_1}+\alpha_{a_2})+\pi\xi+2\pi\epsilon+\eta|\omega_2\omega_3)S_2(i\epsilon(\alpha_{a_1}-\alpha_{a_2})+\pi(\xi+1)|\omega_2\omega_3)} \\
\times \ & \prod_{a\in A}\prod_{\epsilon=\pm}\frac{\Gamma_2(i\epsilon\alpha_a+i\mu_\epsilon-\frac{\pi}{2}+\pi\epsilon|\omega_2\omega_3)\Gamma_2(-i\epsilon\alpha_a+\eta-\frac{\pi}{2}-\pi\epsilon+i\mu_\epsilon|\omega_2\omega_3)}{\Gamma_2(i\epsilon\alpha_a-i\mu_\epsilon+\pi\xi+\frac{\pi}{2}+\pi\epsilon|\omega_2\omega_3)\Gamma_2(-i\epsilon\alpha_a+\eta+\pi\xi+\frac{\pi}{2}-\pi\epsilon-i\mu_\epsilon|\omega_2\omega_3)} \\
\times \ & \prod_{a\in A}\prod_{\epsilon=\pm}\sqrt{\frac{\Gamma_2(2i\epsilon\alpha_a+2\pi\epsilon+\pi|\omega_2,2\omega_3)\Gamma_2(2i\epsilon\alpha_a+2\pi\epsilon+\pi(\xi+1)|\omega_2,2\omega_3)}{\Gamma_2(2i\epsilon\alpha_a+2\pi\epsilon|\omega_2,2\omega_3)\Gamma_2(2i\epsilon\alpha_a+2\pi\epsilon+\pi\xi|\omega_2,2\omega_3)}} \\
\times \ & \prod_{a\in A}\prod_{\epsilon=\pm}\sqrt{\frac{\Gamma_2(2i\epsilon\alpha_a+2\pi\epsilon+2\eta+\pi|\omega_2,2\omega_3)\Gamma_2(2i\epsilon\alpha_a+2\pi\epsilon+2\eta+\pi(\xi+1)|\omega_2,2\omega_3)}{\Gamma_2(2i\epsilon\alpha_a+2\pi\epsilon+2\eta|\omega_2,2\omega_3)\Gamma_2(2i\epsilon\alpha_a+2\pi\epsilon+2\eta+\pi\xi|\omega_2,2\omega_3)}}.
\end{aligned}
\tag{4.27}
$$

The magnetization on a site $m$ is given by

$$
\langle\sigma_m^z\rangle=\frac{\displaystyle\sum_{\epsilon_1,\cdots,\epsilon_N=\pm}\epsilon_m P_{\epsilon_1,\cdots,\epsilon_N;\epsilon_N,\cdots,\epsilon_1;0}(\{0\}|\{0\})}{\displaystyle\sum_{\epsilon_1,\cdots,\epsilon_N=\pm}P_{\epsilon_1,\cdots,\epsilon_N;\epsilon_N,\cdots,\epsilon_1;0}(\{0\}|\{0\})}. \tag{4.28}
$$

*Note. In paper [9] the authors considered the following vacuum expectation value for finite XXZ chain with double boundaries at massive regime,*

$$
\frac{\langle vac|e^F\Phi_{\epsilon_1^*}(\zeta_1^*)\cdots\Phi_{\epsilon_N^*}(\zeta_N^*)e^{F^*}e^G\Phi_{\epsilon_N}(\zeta_N)\cdots\Phi_{\epsilon_1}(\zeta_1)e^{G^*}|vac\rangle}{\langle vac|e^Fe^{F^*}e^Ge^{G^*}|vac\rangle},
$$

*which is free from a difficulty of divergence. However it's physical meaning is not clear.*

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# A Vertex Operators

Here we summarize the bosonizations of the vertex operators [8].

Let us set free bosons $b(t)(t \in \mathbb{R})$ which satisfy

$$[b(t), b(t')] = \frac{\mathrm{sh}\left(\frac{\pi t}{2}\right)\mathrm{sh}(\pi t)\mathrm{sh}\frac{\pi t\xi}{2}}{t\mathrm{sh}\frac{\pi t(\xi+1)}{2}}\delta(t+t'). \tag{A.1}$$

Let us set $a(t)$ by

$$b(t)\mathrm{sh}\frac{\pi t(\xi+1)}{2} = a(t)\mathrm{sh}\frac{\pi t\xi}{2}. \tag{A.2}$$

The bosonization of the type-I vertex operators is given by

$$\Phi_+(\beta) = U(\beta), \tag{A.3}$$

$$\begin{aligned}\Phi_-(\beta) &= \int_{C_I} d\alpha :U(\beta)\bar{U}(\alpha): \\ &\times \Gamma\left(\frac{i(\alpha-\beta)}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right)\Gamma\left(-\frac{i(\alpha-\beta)}{\pi(\xi+1)}+\frac{1}{2(\xi+1)}\right),\end{aligned} \tag{A.4}$$

where we have set

$$U(\alpha) =: \exp\left(-\int_{-\infty}^{\infty}\frac{b(t)}{\mathrm{sh}\pi t}e^{i\alpha t}dt\right):, \quad \bar{U}(\alpha) =: \exp\left(\int_{-\infty}^{\infty}\frac{b(t)}{\mathrm{sh}\frac{\pi}{2}t}e^{i\alpha t}dt\right):. \tag{A.5}$$

The bosonization of the type-II vertex operators is given by

$$\Psi_+(\beta) = V(\beta), \tag{A.6}$$

$$\begin{aligned}\Psi_-(\beta) &= \int_{C_{II}} d\alpha :V(\beta)\bar{V}(\alpha): \\ &\times \Gamma\left(\frac{i(\alpha-\beta)}{\pi\xi}-\frac{1}{2\xi}\right)\Gamma\left(-\frac{i(\alpha-\beta)}{\pi\xi}+-\frac{1}{2\xi}\right),\end{aligned} \tag{A.7}$$

where we have set

$$V(\alpha) =: exp\left(\int_{-\infty}^{\infty} \frac{a(t)}{\text{sh}\pi t} e^{i\alpha t} dt\right) :, \quad \bar{V}(\alpha) =: exp\left(-\int_{-\infty}^{\infty} \frac{a(t)}{\text{sh}\frac{\pi}{2} t} e^{i\alpha t} dt\right) : . \tag{A.8}$$

Here the integration contours are chosen as follows. The contour $C_I$ is $(-\infty, \infty)$. The poles

$$\alpha - \beta = \frac{\pi i}{2} + n\pi(\xi + 1)i, \quad (n \in \mathbb{N}) \tag{A.9}$$

of $\Gamma\left(\frac{i(\alpha-\beta)}{\pi(\xi+1)} + \frac{1}{2(\xi+1)}\right)$ are above $C_I$ and the poles

$$\alpha - \beta = -\frac{\pi i}{2} - n\pi(\xi + 1)i, \quad (n \in \mathbb{N}) \tag{A.10}$$

of $\Gamma\left(-\frac{i(\alpha-\beta)}{\pi(\xi+1)} + \frac{1}{2(\xi+1)}\right)$ are below $C_I$. The contour $C_{II}$ is $(-\infty, \infty)$ except that the poles

$$\alpha - \beta = -\frac{\pi i}{2} + n\pi\xi i, \quad (n \in \mathbb{N}) \tag{A.11}$$

of $\Gamma\left(\frac{i(\alpha-\beta)}{\pi\xi} - \frac{1}{2\xi}\right)$ are above $C_{II}$ and the poles

$$\alpha - \beta = \frac{\pi i}{2} - n\pi\xi i, \quad (n \in \mathbb{N}) \tag{A.12}$$

of $\Gamma\left(-\frac{i(\alpha-\beta)}{\pi\xi} - \frac{1}{2\xi}\right)$ are below $C_{II}$.

# B Multi Gamma functions

Here we summarize the multiple gamma and the multiple sine functions.

Let us set the functions $\Gamma_1(x|\omega), \Gamma_2(x|\omega_1, \omega_2)$ and $\Gamma_3(x|\omega_1, \omega_2, \omega_3)$ by

$$\log\Gamma_1(x|\omega) + \gamma B_{11}(x|\omega) = \int_C \frac{dt}{2\pi i t} e^{-xt} \frac{\log(-t)}{1 - e^{-\omega t}}, \tag{B.1}$$

$$\log\Gamma_2(x|\omega_1, \omega_2) - \frac{\gamma}{2} B_{22}(x|\omega_1, \omega_2) = \int_C \frac{dt}{2\pi i t} e^{-xt} \frac{\log(-t)}{(1 - e^{-\omega_1 t})(1 - e^{-\omega_2 t})}, \tag{B.2}$$

$$\log\Gamma_3(x|\omega_1, \omega_2, \omega_3) + \frac{\gamma}{3!} B_{33}(x|\omega_1, \omega_2, \omega_3) = \int_C \frac{dt}{2\pi i t} e^{-xt} \frac{\log(-t)}{(1 - e^{-\omega_1 t})(1 - e^{-\omega_2 t})(1 - e^{-\omega_3 t})}, \tag{B.3}$$

where the functions $B_{jj}(x)$ are the multiple Bernoulli polynomials defined by

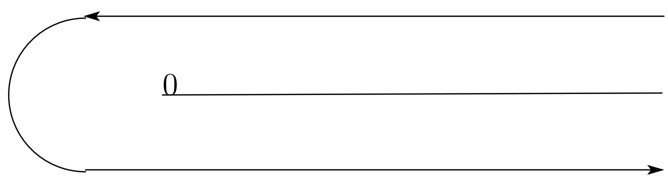
$$\frac{t^r e^{xt}}{\prod_{j=1}^r (e^{\omega_j t} - 1)} = \sum_{n=0}^{\infty} \frac{t^n}{n!} B_{r,n}(x|\omega_1 \cdots \omega_r), \tag{B.4}$$

more explicitly

$$B_{11}(x|\omega) = \frac{x}{\omega} - \frac{1}{2}, \tag{B.5}$$

$$B_{22}(x|\omega) = \frac{x^2}{\omega_1\omega_2} - \left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right) x + \frac{1}{2} + \frac{1}{6}\left(\frac{\omega_1}{\omega_2} + \frac{\omega_2}{\omega_1}\right). \tag{B.6}$$

Here $\gamma$ is Euler's constant, $\gamma = \lim_{n\to\infty}(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n)$.
Here the contor of integral is given by



**Contour** $C$

Let us set

$$S_1(x|\omega) = \frac{1}{\Gamma_1(\omega - x|\omega)\Gamma_1(x|\omega)}, \tag{B.7}$$

$$S_2(x|\omega_1,\omega_2) = \frac{\Gamma_2(\omega_1+\omega_2 - x|\omega_1,\omega_2)}{\Gamma_2(x|\omega_1,\omega_2)}, \tag{B.8}$$

$$S_3(x|\omega_1,\omega_2,\omega_3) = \frac{1}{\Gamma_3(\omega_1+\omega_2+\omega_3 - x|\omega_1,\omega_2,\omega_3)\Gamma_3(x|\omega_1,\omega_2,\omega_3)} \tag{B.9}$$

We have

$$\Gamma_1(x|\omega) = e^{(\frac{x}{\omega}-\frac{1}{2})\log\omega}\frac{\Gamma(x/\omega)}{\sqrt{2\pi}},\ S_1(x|\omega) = 2\sin(\pi x/\omega), \tag{B.10}$$

$$\frac{\Gamma_2(x+\omega_1|\omega_1,\omega_2)}{\Gamma_2(x|\omega_1,\omega_2)} = \frac{1}{\Gamma_1(x|\omega_2)},\ \frac{S_2(x+\omega_1|\omega_1,\omega_2)}{S_2(x|\omega_1,\omega_2)} = \frac{1}{S_1(x|\omega_2)},\ \frac{\Gamma_1(x+\omega|\omega)}{\Gamma_1(x|\omega)} = x. \tag{B.11}$$

$$\frac{\Gamma_3(x+\omega_1|\omega_1,\omega_2,\omega_3)}{\Gamma_3(x|\omega_1,\omega_2,\omega_3)} = \frac{1}{\Gamma_2(x|\omega_2,\omega_3)},\ \frac{S_3(x+\omega_1|\omega_1,\omega_2,\omega_3)}{S_3(x|\omega_1,\omega_2,\omega_3)} = \frac{1}{S_2(x|\omega_2,\omega_3)}. \tag{B.12}$$

$$\log S_2(x|\omega_1\omega_2) = \int_C \frac{\mathrm{sh}(x - \frac{\omega_1+\omega_2}{2})t}{2\mathrm{sh}\frac{\omega_1 t}{2}\mathrm{sh}\frac{\omega_2 t}{2}} \log(-t)\frac{dt}{2\pi i t},\ (0 < \mathrm{Re}x < \omega_1 + \omega_2). \tag{B.13}$$

$$S_2(x|\omega_1\omega_2) = \frac{2\pi}{\sqrt{\omega_1\omega_2}}x + O(x^2), \quad (x \to 0). \tag{B.14}$$

$$S_2(x|\omega_1\omega_2)S_2(-x|\omega_1\omega_2) = -4\sin\frac{\pi x}{\omega_1}\sin\frac{\pi x}{\omega_2}. \tag{B.15}$$